| Binomial Approximation | Taylor's Theorem |
|---|---|

In physical sciences we often have powers that can be approximated.

Why approximate?
- Ease Derivations
- Reduce Computational Workload
- Simplify Equations

## The Binomial Approximation

Lets look at a few powers of numbers close to 1.

$$(1+x)^1 = 1 + 1x$$

$$(1+x)^2 = (1+x)(1+x) = 1 + 2x + x^2$$

$$(1+x)^3 = (1+x)(1+x)^2 = (1+x)(1+2x+x^2)$$

$$= 1 + 2x + x^2 + x + 2x^2 + x^3$$

$$= 1 + 3x + 3x^2 + x^3$$

We see a pattern emerging

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \dots + x^\alpha$$

$\alpha$ can be positive or negative and does NOT have to be an integer

Consider a small $x$, so $|x| \ll 1$

We know $|x^3| \ll |x^2| \ll |x|$ * To do this multiply both sides of the inequality above by $|x|$

This means for a small $x$ we can say:

$$\Longrightarrow \boxed{(1+x)^\alpha \cong 1 + \alpha x}$$

**Example**

$$\left(1+\frac{1}{100}\right)^{8.2} = 1.0850 \text{ (from calculator)}$$

Use Binomial Approximation $\left(1+\frac{1}{100}\right)^{8.2} \cong 1 + 8.2(0.01) = 1.082$

$$\text{Error} = \frac{1.082 - 1.0850}{1.0850} \times 100\% = -0.28\% < 1\% \text{ Not Bad!}$$

**Example**

$$\left(10+\frac{1}{100}\right)^{7.6} = 40{,}114{,}278.8$$

$$\left(10+\frac{1}{100}\right)^{7.6} = 10^{7.6}\left(1+\frac{1}{1000}\right)^{7.6} \cong 10^{7.6}\cdot\left(1+7.6\cdot\frac{1}{1000}\right) = 40{,}113{,}278.5$$

(error << 1%).

**Example**

$$(1+70)^{3.4} = 1{,}969{,}148.12$$

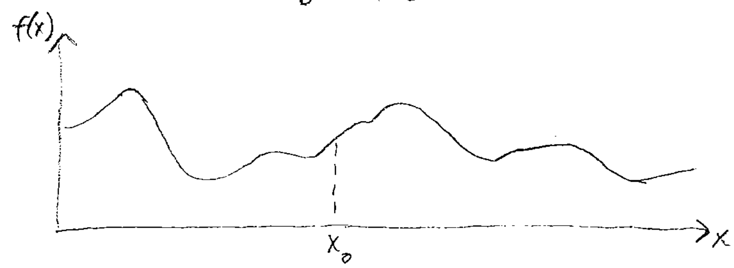
$$(1+70)^{3.4} = (70+1)^{3.4} = 70^{3.4}\left(1+\frac{1}{70}\right)^{3.4}$$

$$70^{3.4}\left(1+\frac{1}{70}\right)^{3.4} \cong 70^{3.4}\cdot\left(1+\frac{3.4}{70}\right) = 1{,}967{,}575.27$$

(error << 1%)

**Taylor's Theorem**

Pick some arbitrary function:




If you look at any VERY SMALL section of the curve around any point $x_0$ it looks like a straight line. Larger sections begin to resemble higher order functions like quadratic ($x^2$), cubic ($x^3$), and quartic ($x^4$) functions.

Expanding $f(x)$ about $x_0$:

$$f(x) = f(x_0) + \left.\frac{df}{dx}\right|_{x_0}(x-x_0) + \left.\frac{d^2f}{dx^2}\right|_{x_0}\frac{(x-x_0)^2}{2!} + \left.\frac{d^nf}{dx^n}\right|_{x_0}\frac{(x-x_0)^n}{n!} + \cdots$$

**Example**

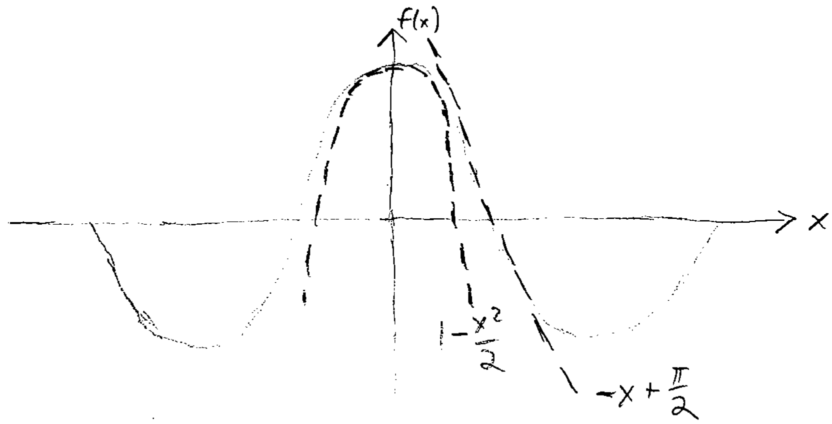
Expand $f(x) = \cos(x)$ about $x_0$

$$\cos(x) = \cos(x_0) - \sin(x_0)(x-x_0) - \cos(x_0)\frac{(x-x_0)^2}{2!} + \sin(x_0)\frac{(x-x_0)^3}{3!} + \cos(x_0)\frac{(x-x_0)^4}{4!} + \cdots$$

Lets consider $X_0 = \emptyset$

$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

For $|x| \ll 1$ (near $x_0 = \emptyset$) $\cos(x) \approx 1 - \frac{x^2}{2}$




* Remember Euler's Formula? We could get there from here
But we won't.

Distance Between Points

Finding Vectors

## Finding the Distance Between Points

$P_1 = (x_1, y_1) \qquad P_2 = (x_2, y_2)$

$$D(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

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$P_3 = (x_3, y_3, z_3) \qquad P_4 = (x_4, y_4, z_4)$

$$D(P_3, P_4) = \sqrt{(x_4 - x_3)^2 + (y_4 - y_3)^2 + (z_4 - z_3)^2}$$

## Finding a Vector Between Two Points

$$\overrightarrow{P_1P_2} = \langle \underset{\hat{i}}{\underset{\uparrow}{x_2 - x_1}} , \underset{\hat{j}}{\underset{\uparrow}{y_2 - y_1}} \rangle$$

$$\overrightarrow{P_3P_4} = \langle \underset{\hat{i}}{\underset{\uparrow}{x_2 - x_1}} , \underset{\hat{j}}{\underset{\uparrow}{y_2 - y_1}} , \underset{\hat{k}}{\underset{\uparrow}{z_2 - z_1}} \rangle$$

# Vector / Calculus Review

Vector – geometric object with both a magnitude and direction commonly noted with an arrow.

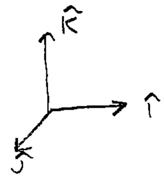

Magnitude is $|\vec{A}| \equiv A$

Unit Vector – Any vector which has a magnitude of 1. commonly denoted with a "hat"

So a unit vector in the direction of $\vec{A}$ above is $\hat{a} = \frac{\vec{A}}{|\vec{A}|}$

Think of a vector in normal 3D space.

$$\vec{A} = \underbrace{a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}}_{\text{magnitudes}}$$

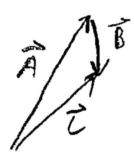



$$|\vec{A}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

## Vector Addition

"Tip to tail"

$$\vec{C} = \vec{A} + \vec{B}$$



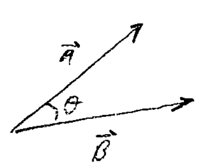

Properties of Vector Addition:

- Commutative $\vec{A} + \vec{B} = \vec{B} + \vec{A}$
- Associative $(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$
- Distributive $c(\vec{A} + \vec{B}) = c\vec{A} + c\vec{B}$ or $(b + c)\vec{A} = b\vec{A} + c\vec{A}$

## Dot Product

- Product of vector magnitudes and the cosine of the angle between them. "How parallel are they?"

$$\vec{A} \cdot \vec{B} = AB\cos\theta$$




For $\vec{A} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ $\quad \vec{B} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

$$\vec{A} \cdot \vec{B} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

When is the dot product $\emptyset$? When is it maximized?
Is the output a scalar or vector?

## Cross Product

Results in a vector which is perpendicular to both of the vectors being multiplied and normal to the plane containing them.

$$|\vec{A} \times \vec{B}| = AB\sin\theta$$

To determine direction point RH fingers in direction of $\vec{A}$ and curl them towards $\vec{B}$. The thumb is the resultant direction.

<demonstrate RH Rule>

### Examples

$\hat{\imath} \times \hat{\jmath} = \hat{k}$ $\qquad\qquad$ $\hat{k} \times \hat{k} = 0$

$\hat{k} \times \hat{\imath} = \hat{\jmath}$

$\hat{\jmath} \times \hat{\imath} = -\hat{k}$

### Evaluating the Cross Product in Matrix Notation

$$\vec{A} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k} \qquad \vec{B} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

2 ways to approach this determinant.

1) $$\vec{A} \times \vec{B} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{\imath} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{k}$$

2) $$\vec{A} \times \vec{B} = \left|\begin{matrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix}\right| \begin{matrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & \\ b_1 & b_2 & \end{matrix}$$

Both lead to $\vec{A} \times \vec{B} = \hat{\imath}\, a_2 b_3 + \hat{\jmath}\, a_3 b_1 + \hat{k}\, a_1 b_2 - \hat{\imath}\, a_3 b_2 - \hat{\jmath}\, a_1 b_3 - \hat{k}\, a_2 b_1$

# Gradient, Divergence + Curl

The gradient of a vector field is represented by "del" $\nabla$

$f(x,y,z)$ is our function of interest

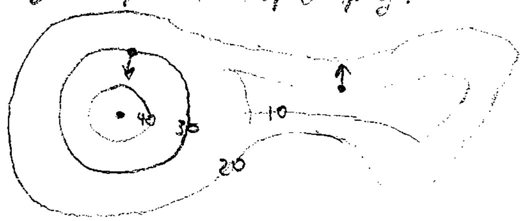
$$\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$$

$$\nabla f = \text{grad } f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}$$

The gradient vector always points toward <u>HIGHER</u> values

<u>Example</u>

Draw the gradient vector at the given points on the following map of topography.




<u>Example</u>

Find the gradient of the following function:

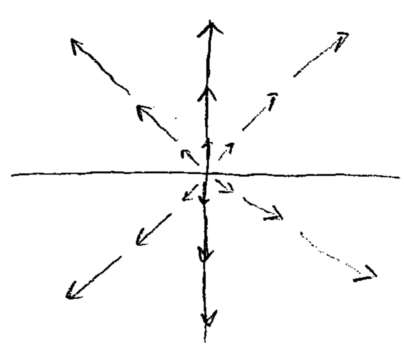
$$f(x,y,z) = 2x^3 + 3xy + y^2 + 6\cos(z)$$

$$\nabla f = \langle (6x^2 + 3y), (3x + 2y), (-6\sin(z)) \rangle$$

<u>Divergence</u>

$$\text{div } \vec{A} = \nabla \cdot \vec{A} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z} \quad \text{(scalar result)}$$

Imagine fluid exploding from a central point



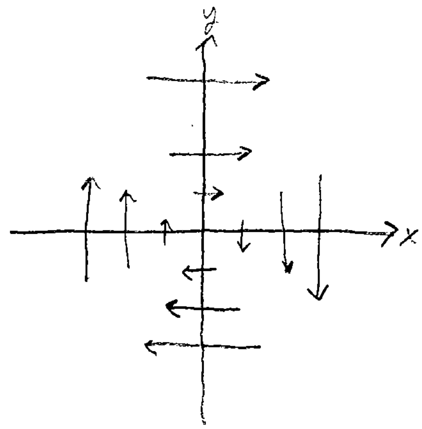
$\text{div } \vec{A} > 0$

# Curl

Describes the rotation of a vector field

$$\text{curl}\,\vec{A} = \nabla \times \vec{A} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} = \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right)\hat{\imath} + \left(\frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x}\right)\hat{\jmath}$$

$$+ \left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y}\right)\hat{k}$$




* Imagine placing a pinwheel Anywhere in this flow

what is the curl?

NEGATIVE $\hat{k}$ (use RH rule)

what if all vectors were the same magnitude?